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Elements of $\mathscr{D}'^{(M_p)}_{L^s}$ and $\mathscr{D}'^{\{M_p\}}_{L^s}$ as Boundary Values of Holomorphic Functions

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We give representations of elements belonging to $\mathscr{D}'^{(M_p)}_{L^s}$ and $\mathscr{D}'^{\{M_p\}}_{L^s}$, $s \in (1, \infty)$, where M_p is a non-quasianalytic sequence, as boundary values of holomorphic functions which satisfy appropriate estimates on the boundary of their domains. Conversely, we prove that the same estimates on holomorphic functions imply their $\mathscr{D}'^{(M_p)}_{L^s}$- and $\mathscr{D}'^{\{M_p\}}_{L^s}$-boundary values. With the stronger assumptions on M_p we obtain that appropriate L^∞, resp. L^1, estimates on holomorphic functions imply their boundary values in $\mathscr{D}'^{(M_p)}_{L^\infty}$ and $\mathscr{D}'^{\{M_p\}}_{L^\infty}$, resp. $\mathscr{D}'^{(M_p)}_{L^1}$ and $\mathscr{D}'^{\{M_p\}}_{L^1}$. © 1996 Academic Press, Inc.

INTRODUCTION

The characterization of holomorphic function spaces whose elements have boundary values in spaces of distributions, ultradistributions, infra-hyperfunctions, and, conversely, the boundary value representation of elements in quoted generalized function spaces by holomorphic functions, has a long history; see [5, 12, 4, 3, 6, 10]. Let us call the characterizations depicted above the boundary value characterization of corresponding generalized function spaces.

The complete boundary value characterization for the spaces $\mathscr{D}'^{(M_p)}$, $\mathscr{D}'^{\{M_p\}}$, $\mathscr{E}'^{(M_p)}$, $\mathscr{E}'^{\{M_p\}}$, of ultradistributions, resp. infra-hyperfunctions, which are related to a non-quasianalytic, resp. quasianalytic, sequence M_p are given in [6]. We have studied in [7–9] spaces $\mathscr{D}'^{(M_p)}_{L^s}$ and $\mathscr{D}'^{\{M_p\}}_{L^s}$, $s \geq 1$, related to a non-quasianalytic sequence M_p. By using the Fourier transformation and the results of Komatsu [3] we have given in [7] for $s = 2$ and in [1] for $s \in [1, 2]$ the boundary value characterization for Beurling type

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spaces, while in [8], for $s \geq 1$, we have given a partial characterization by considering all the spaces $\mathscr{D}'^{(M_p)}_{L^s}$, $s \geq 1$, as subspaces of $\mathscr{D}'^{(M_p)}_{L^\infty}$. In the mean time the paper [2] has appeared which was written after this one.

By using a simple and powerful method of [6] based on almost analytic extension and Stokes' theorem, we give in this paper the complete boundary value characterization for $\mathscr{D}'^{(M_p)}_{L^s}$ and $\mathscr{D}'^{\{M_p\}}_{L^s}$ spaces, $s > 1$, related to a non-quasianalytic sequence M_p. In the case $s = \infty$ and $s = 1$ we could not use the same method as for $s > 1$. Because of that we are forced to use the method of Komatsu [3] and to assume, instead of (3), condition (4) which implies that an appropriate L^∞, resp. L^1, estimate on a holomorphic function implies that the corresponding boundary value belongs to the ultradistribution space with the subindex L^∞, resp. L^1.

1. NOTATION AND NOTIONS

As usual, $\mathbf{N}$, $\mathbf{R}$, $\mathbf{C}$ denote the sets of natural, real, and complex numbers; $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$. The norm in L^p, $p \in [1, \infty]$ is denoted by $\| \ \|_p$. We denote by $\psi|\mathbf{R}$ the restriction of ψ on $\mathbf{R}$.

Assume that M_p is a strictly increasing sequence of positive numbers such that $M_0 = 1$. Put $M_p^* = M_p/p!$, $m_p = M_p/M_{p-1}$, $m_p^* = m_p/p$; $p \in \mathbf{N}$. As in [6], we assume

$$1 \leq m_p^* \leq m_{p+1}^*, \ p \in \mathbf{N}, \qquad \text{and} \qquad \lim_{p \to \infty} m_p^* = \infty; \tag{1}$$

$$\sup\left\{ \frac{m_{2p}}{m_p}; \ p \in \mathbf{N} \right\} < \infty; \tag{2}$$

$$\sum_{p=1}^{\infty} 1/m_p < \infty. \tag{3}$$

In the fourth section, instead of (3) we assume the stronger condition

$$\sum_{q=p}^{\infty} M_q/M_{q+1} \leq ApM_p/M_{p+1}, \qquad p \in \mathbf{N}. \tag{4}$$

Note that (1) implies (M.1) of [3], (2) is equivalent to (M.2) of [3], and (3) and (4) are denoted in [3] by (M.3)′ and (M.3). If (3) holds, then we say that this sequence is a non-quasianalytic one, otherwise we say that it is a quasianalytic sequence. Note, (3) implies $p! \prec M_p$ [3]. In general, for positive sequences N_p and M_p satisfying (M.1) $M_p \prec N_p$ means:

for every $L > 0$, there is $C > 0$ such that $M_p \leq CL^pN_p$, $p \in \mathbf{N}_0$ [3].

As in [3], we put

$$M(\rho) = \sup_{p \in \mathbf{N}_0} \left\{ \ln \frac{\rho^p}{M_p} \right\}, \qquad M^*(\rho) = \sup_{p \in \mathbf{N}_0} \left\{ \ln \frac{\rho^p}{M_p^*} \right\}, \qquad \rho > 0,$$

$$m^*(\lambda) = \max\{p; m_p^* \le \lambda\}, \qquad \lambda > 0.$$

For the properties of the sequence M_p and the functions M and M^* we refer to [3, 6].

The spaces $\mathscr{D}_{L^r}^{(M_p)}$ and $\mathscr{D}_{L^r}^{\{M_p\}}$, $r \ge 1$, are defined as follows. Let $h > 0$. We put

$$\mathscr{D}_{L^r,h}^{M_p} = \left\{ \phi \in C^\infty(\mathbf{R}); \|\phi\|_{L^r,h} = \sup_{p \in \mathbf{N}_0} \left\{ \frac{h^p}{M_p} \|\phi^{(p)}\|_{L^r} < \infty \right\} \right\};$$

$$\mathscr{D}_{L^r}^{(M_p)} = \operatorname{proj} \lim_{h \to \infty} \mathscr{D}_{L^r,h}^{M_p}; \qquad \mathscr{D}_{L^r}^{\{M_p\}} = \operatorname{ind} \lim_{h \to 0} \mathscr{D}_{L^r,h}^{M_p}.$$

$\mathscr{D}_{L^r}^*$ is a common notation for both spaces. $\mathscr{B}^*$ is the completion of $\mathscr{D}^*$ in $\mathscr{D}_{L^\infty}^*$. Their strong duals are $\mathscr{D}_{L^s}'^*$, $s \in [1, \infty)$, where $s = r/(r-1)$ (for $r = 1$, $s = \infty$). The strong dual of $\dot{\mathscr{B}}^*$ is $\mathscr{D}_{L^1}'^*$. For the properties of such spaces we refer to [1, 9].

Let $r \ge r' \ge 1$. We have

$$\mathscr{D}^* \hookrightarrow \mathscr{D}_{L^1}^* \hookrightarrow \mathscr{D}_{L^r}^* \hookrightarrow \mathscr{D}_{L^{r'}}^* \hookrightarrow \mathscr{E}^*, \qquad \dot{\mathscr{B}}^* \hookrightarrow \mathscr{E}^*,$$

where "$A \hookrightarrow B$" means that A is a dense subspace of B and that the inclusion mapping is continuous. $\mathscr{D}^*$ and $\mathscr{E}^*$ are the common notation for the spaces of Beurling and Roumieu ultradifferentiable functions $\mathscr{D}^{(M_p)}$, $\mathscr{D}^{\{M_p\}}$ and $\mathscr{E}^{(M_p)}$, $\mathscr{E}^{\{M_p\}}$. We refer to [3] for these spaces.

An operator, formally of the form $\sum_{\alpha=0}^\infty a_\alpha D^\alpha$ ($a_\alpha \in \mathbf{C}$), is called an ultradifferential operator of class (M_p), resp. of class $\{M_p\}$, if for some $k > 0$ and $C > 0$, resp. for every $k > 0$ there is $C > 0$, there holds

$$|a_\alpha| \le Ck^\alpha / M_\alpha, \qquad \alpha \in \mathbf{N}_0 \text{ [3]}. \tag{5}$$

Let us recall [9] that $f \in \mathscr{D}_{L^s}'^{(M_p)}$, resp. $f \in \mathscr{D}_{L^s}'^{\{M_p\}}$, $s \in (1, \infty]$, iff there is a sequence f_p, $p \in \mathbf{N}_0$, of functions in L^s such that

$$f = \sum_{p=0}^\infty f_p^{(p)} \text{ in the sense of convergence in } \mathscr{D}_{L^s}'^{(M_p)}, \text{ resp. } \mathscr{D}_{L^s}'^{\{M_p\}}, \text{ and}$$

$$\sum_p^\infty \frac{M_p}{k^p} \|f_p\|_{L^r} < \infty \text{ for some } k > 0, \text{ resp.}$$

$$\sum_p \frac{M_p}{k^p} \|f_p\|_{L^r} < \infty, \text{ for every } k > 0. \tag{6}$$

Denote by $H_{L^s}^{(M_p)}$, resp. $H_{L^s}^{\{M_p\}}$, the space of functions f holomorphic in $\psi_0 \setminus \mathbf{R}$, where $\psi_0 = \{x+iy; x \in \mathbf{R}, |y| < \delta_0\}$, $\delta_0 = \delta_0(f) > 0$, which satisfy the following estimate: For some $k > 0$ and some $C > 0$, resp. for every $k > 0$ there is $C > 0$, such that

$$\| f(\cdot + iy) \|_{L^s} \leq C e^{M^*(k/|y|)}, \qquad |y| < \delta_0, \, y \neq 0.$$

The common notation for both spaces is $H_{L^s}^*$.

We denote by H_{L^s} the space of functions f which are holomorphic in ψ_0, such that for some $C > 0$

$$\| f(\cdot + iy) \|_{L^s} < C, \qquad |y| < \delta_0.$$

2. PRELIMINARY LEMMAS

By using the same method as in [6, 2.2. Proposition] (and the Minkovski inequality) one can prove the following lemma:

LEMMA 1. *Let $h > 0$ be given. There is $H > 0$ such that for every $\varphi \in \mathscr{D}_{L^r,h}^{M_p}$ there are $\phi \in C^1(\mathbf{C})$ and $C > 0$ such that $\phi|\mathbf{R} = \varphi$ and*

$$\sup_{y \in \mathbf{R}} \left\{ e^{M^*(hH/|y|)} \left\| \frac{\partial}{\partial \bar{z}} \phi(\cdot + iy) \right\|_{L^r}, \| \phi^{(i)}(\cdot + iy) \|_{L^r}, \, i = 0, 1 \right\} < C \|\varphi\|_{L^r,h}.$$

(If $y = 0$, then $(\partial/\partial\bar{z})\phi(x) = 0$.)

We remark that in Lemma 1, we add the estimate for $\phi'(\cdot + iy)$ in order to have a symmetric assertion to the assertion of Lemma 4.

For the main assertions we need the following three lemmas.

LEMMA 2. *Let F be a holomorphic function on $\mathbf{C} \setminus \mathbf{R}$ such that, in the (M_p) case, there are $k > 0$ and $C > 0$, resp. in the $\{M_p\}$ case, for every $k > 0$ there is $C > 0$, such that*

$$\|F(\cdot + iy)\|_{L^s} \leq C e^{M^*(k/|y|)}, \qquad y \neq 0.$$

Then, for every compact set $K \subset \mathbf{R}$ there are $p > 0$ and $B > 0$, resp. for every $p > 0$ there is $B > 0$, such that

$$\sup_{x \in K} \{|F(x+iy)|\} \leq B e^{M^*(p/|y|)}, \qquad y \neq 0.$$

Proof. We shall prove the assertion only for the (M_p)-case since the proof for the $\{M_p\}$-case is similar.

Let $\alpha \in \mathscr{D}^{(M_p)}$, supp $\alpha \subset [-a, a]$, and $\alpha \equiv 1$ in a neighborhood of K. For $x \in K$ and $y \neq 0$ we have

$$F(x+iy) = \alpha(x) F(x+iy) = \int_{-\infty}^{x} (\alpha(t) F(t+iy))' \, dt.$$

Let $K_t = \{z | \cdot |z - t - iy| = |y|/4\}$, $x \in K$, and $s = r/(r-1)$. By using Cauchy's formula and Hölder's inequality we have (with suitable constants)

$$\begin{aligned}
&|F(x+iy)| \\
&\quad \le C_1\left(\int_{-a}^{a} |\alpha'(t)| \left|\int_{z\in K_t} \frac{F(z)}{z-t-iy}\,dz\right| dt \right.\\
&\qquad\qquad \left. + \int_{-a}^{a} |\alpha(t)| \left|\int_{z\in K_t} \frac{F(z)\,dz}{(z-t-iy)^2}\right| dt\right)\\
&\quad \le C_1\left[\left(\int_{-a}^{a} |\alpha'(t)|^r\,dt\right)^{1/r}\left(\int_{-a}^{a}\left|\int_{z\in K_t} \frac{F(z)}{z-t-iy}\,dz\right|^s dt\right)^{1/s}\right.\\
&\qquad\qquad \left. + \left(\int_{-a}^{a} |\alpha(t)|^r\,dt\right)^{1/r}\left(\int_{-a}^{a}\left|\int_{z\in K_t} \frac{F(z)}{(z-t-iy)^2}\,dz\right|^s dt\right)^{1/s}\right]\\
&\quad \le C_2\left[\left(\int_{-a}^{a}\left|\int_0^{2\pi} F\left(t+iy+\frac{|y|}{4}e^{i\varphi}\right)d\varphi\right|^s dt\right)^{1/s}\right.\\
&\qquad\qquad \left. + \frac{1}{|y|}\left(\int_{-a}^{a}\left|\int_0^{2\pi} \frac{F\big(t+iy+(|y|/4)e^{i\varphi}\big)}{e^{i\varphi}}\,d\varphi\right|^s dt\right)^{1/s}\right]\\
&\quad \le C_2\left[\left(\int_{-a}^{a}\left(\int_0^{2\pi} 1\,d\varphi\right)^{s/r}\int_0^{2\pi}\left|F\left(t+iy+\frac{|y|}{4}\right)\right|^s d\varphi\,dt\right)^{1/s}\right.\\
&\qquad\qquad + \frac{1}{|y|}\left(\int_{-a}^{a}\left(\int_0^{2\pi} 1\,d\varphi\right)^{s/r}\right.\\
&\qquad\qquad\qquad \left.\left. \times\int_0^{2\pi}\left|\frac{F\big(t+iy+(|y|/4)e^{i\varphi}\big)}{e^{i\varphi}}\right|^s d\varphi\,dt\right)^{1/s}\right]\\
&\quad \le C_3\left[\left(\int_{-a}^{a}\int_0^{2\pi}\left|F\left(t+iy+\frac{|y|}{4}e^{i\varphi}\right)\right|^s d\varphi\,dt\right)^{1/s}\right.\\
&\qquad\qquad \left. + \frac{1}{|y|}\left(\int_{-a}^{a}\int_0^{2\pi}\left|F\left(t+iy+\frac{|y|}{4}e^{i\varphi}\right)\right|^s d\varphi\,dt\right)^{1/s}\right]
\end{aligned}$$

$$\leq C_4\left(1+\frac{1}{|y|}\right)\left(\int_0^{2\pi}\left\|F\left(\cdot+iy+\frac{|y|}{4}e^{i\varphi}\right)\right\|_s^s d\varphi\right)^{1/s}$$
$$\leq C_5\left(1+\frac{1}{|y|}\right)\exp\left(M^*\left(\frac{k}{|y+(|y|/4)\sin\varphi|}\right)\right)$$
$$\leq C_5\left(1+\frac{1}{|y|}\right)\exp\left(M^*\left(\frac{k}{|y|-|y|/4}\right)\right)$$
$$\leq C_5\left(1+\frac{1}{|y|}\right)\exp(M^*(4k/|y|)) \leq C_6\exp(M^*(5k/|y|)).$$

The lemma is proved.

By using Sobolev's lemma one can easily prove the following one.

LEMMA 3. *Let $\varphi \in \mathscr{D}_{L^r}^{(M_p)}$, resp. $\varphi \in \mathscr{D}_{L^r}^{\{M_p\}}$. Then for every compact set $K \subset \mathbf{R}$ and every $h > 0$, resp. for some $h > 0$, there are $C > 0$ and $k > 0$, such that*

$$\sup_{\substack{x\in K\\ p\in\mathbf{N}_0}}\left\{\frac{h^p}{M_p}\left|\varphi^{(p)}(x)\right|\right\} \leq C\|\varphi\|_{k,L^r}.$$

LEMMA 4. *Let $\psi_0 = \{z; |\mathrm{Im}\, z| < \delta_0\}$, $\delta_0 > 0$, $\phi^{(i)}(\cdot + iy) \in L^r$, $i = 0, 1$, $|y| < \delta_0$, and $\phi \in C^1(\psi_0)$. Assume that for every $h > 0$, resp. some $h > 0$,*

$$D_h = \sup_{0<|y|<\delta_0}\left\{\left\|\frac{\partial}{\partial\bar z}\phi(\cdot+iy)\right\|_{L^r}e^{M^*(h/|y|)}, \left\|\phi^{(i)}(\cdot+iy)\right\|_{L^r}, i=0,1\right\} < \infty. \tag{7}$$

Then, $\varphi = \psi|\mathbf{R}$ is in $\mathscr{D}_{L^r}^{(M_p)}$, resp. $\mathscr{D}_{L^r}^{\{M_p\}}$, and for every $h > 0$, resp. for some $h > 0$, there is $C > 0$ such that

$$\|\varphi\|_{h,L^r} \leq CD_h.$$

Proof. We denote $\Gamma_{a,\delta\pm} = \{\zeta; \zeta = t \pm i\delta, |t| < a\}$, $\psi_a = \{\zeta; |\mathrm{Im}\,\zeta| < \delta, |\mathrm{Re}\,\zeta| < a\}$, $\gamma_{a,\pm} = \{\zeta; \zeta = \pm a + it, |t| < \delta\}$, $\Gamma_{\delta\pm} = \{\zeta; \zeta = t \pm i\delta, t \in \mathbf{R}\}$, $\psi = \{\zeta; |\mathrm{Im}\,\zeta| < \delta\}$. This notation will be used later, as well.

Let $x \in \mathbf{R}$, $p \in \mathbf{N}$. By Cauchy's formula, for sufficiently large a, we have

$$\phi^{(p)}(x) = \frac{p!}{2\pi i}\left(\int_{\Gamma_{a,\delta^-}} \frac{\phi(\zeta)\,d\zeta}{(\zeta - x)^{p+1}} - \int_{\Gamma_{a,\delta^+}} \frac{\phi(\zeta)\,d\zeta}{(\zeta - x)^{p+1}} + \int_{\gamma_{a,+}} \frac{\phi(\zeta)\,d\zeta}{(\zeta - x)^{p+1}} - \int_{\gamma_{a,-}} \frac{\phi(\zeta)\,d\zeta}{(\zeta - x)^{p+1}} + \int_{\psi_a} \frac{(\partial/\partial\bar{\zeta})\phi(\zeta)\,d\zeta \wedge d\bar{\zeta}}{(\zeta - x)^{p+1}}\right).$$

Since

$$|\phi(x+iy)| = \left|\int_0^x \phi'(t+iy)\,dt\right| \le |x|^{1/s}\left(\int_{-\infty}^{\infty} |\phi'(t+iy)|^r\,dt\right)^{1/r},$$

(7) implies that for every $p \in \mathbf{N}$,

$$\int_{\gamma_{a\pm}} \frac{\phi(\zeta)\,d\zeta}{(\zeta - x)^{p+1}} \to 0 \qquad \text{as } a \to 0.$$

This implies

$$\phi^{(p)}(x) = \frac{p!}{2\pi i}\left(\int_{\Gamma_{\delta^-}} \frac{\phi(\zeta)\,d\zeta}{(\zeta - x)^{p+1}} - \int_{\Gamma_{\delta^+}} \frac{\phi(\zeta)\,d\zeta}{(\zeta - x)^{p+1}} + \int_{\psi} \frac{(\partial/\partial\bar{\zeta})\phi(\zeta)\,d\zeta \wedge d\bar{\zeta}}{(\zeta - x)^{p+1}}\right)$$
$$= \frac{1}{2\pi i}(I_1 - I_2 + I_3).$$

Let us estimate I_1, I_2, and I_3:

$$|I_1|^r \le p!^r\left(\int_{-\infty}^{\infty} \frac{|\phi(t+x-i\delta)|\,dt}{|t-i\delta|^{p+1}}\right)^r$$
$$\le p!^r \int_{-\infty}^{\infty} \frac{|\phi(t+x-i\delta)|^r\,dt}{|t-i\delta|^{pr}}\left(\int_{-\infty}^{\infty} \frac{dt}{(t^2+\delta^2)^{s/2}}\right)^{r/s}$$
$$\le \frac{Ap!^r}{\delta}\int_{-\infty}^{\infty} \frac{|\phi(t+x-i\delta)|^r\,dt}{|t-i\delta|^{pr}},$$

$$\text{where } A = \left(\int_{-\infty}^{\infty} \frac{dt}{(1+t^2)^{s/2}}\right)^{r/s}.$$

By Hölder's inequality and Fubini's theorem we have

$$\begin{aligned}\int_{-\infty}^{\infty} |I_1|^r\,dx &\le \frac{Ap!^r}{\delta}\int_{-\infty}^{\infty}\frac{dt}{|t-i\delta|^2}\int_{-\infty}^{\infty}\frac{|\phi(t+x-i\delta)|^r}{|t-i\delta|^{pr-2}}\,dx\\ &\le \frac{Ap!^r}{\delta}\int_{-\infty}^{\infty}\frac{dt}{t^2+\delta^2}\int_{-\infty}^{\infty}\frac{|\phi(t+x-i\delta)|^r\,dx}{\delta^{pr-2}}\\ &\le \frac{A\pi p!^r}{2\delta^2}\,\frac{1}{\delta^{pr-2}}D_h^r.\end{aligned}$$

Thus, by using $p!\prec M_p$ we obtain that for suitable $\tilde{A}>0$,

$$\left(\int_{-\infty}^{\infty}|I_1|^r\,dx\right)^{1/r} \le \tilde{A}D_h\frac{p!}{\delta^p e^{M^*(h/\delta)}} \le \tilde{A}D_h h^{-p}M_p.$$

The same inequality holds for $\|I_2\|_{L^r}$. Let us estimate $\|I_3\|_{L^r}$. We have

$$\begin{aligned}&\left|\int\int_{\psi}\frac{(\partial/\partial\bar{\zeta})\phi(\zeta)\,d\zeta\wedge d\bar{\zeta}}{(\zeta-x)^{p+1}}\right|^r\\ &\le\left(\int_{-\infty}^{\infty}\int_{-\delta}^{\delta}\frac{|(\partial/\partial\bar{\zeta})\phi(\xi+i\eta)|\,d\eta\,d\xi}{|\xi+i\eta-x|^{p+1}}\right)^r\\ &=\left(\int_{-\infty}^{\infty}\int_{-\delta}^{\delta}\frac{|(\partial/\partial\bar{\zeta})\phi(\xi+i\eta)|}{|\eta|^{1/s}|\xi+i\eta-x|^{p+1-(2/s)}}\,\frac{|\eta|^{1/s}\,d\eta\,d\xi}{|\xi+i\eta-x|^{2/s}}\right)^r\\ &\le\int_{-\infty}^{\infty}\int_{-\delta}^{\delta}\left(\frac{|(\partial/\partial\bar{\zeta})\phi(\xi+i\eta)|^r\,d\eta\,d\xi}{|\eta|^{r/s}|\xi+i\eta-x|^{(p+1-(2/s))r}}\right)\left(\int_{-\infty}^{\infty}\int_{-\delta}^{\delta}\frac{|\eta|\,d\eta\,d\xi}{|\xi+i\eta-x|^2}\right)^{r/s}\\ &=p!^r\int_{-\infty}^{\infty}\int_{-\delta}^{\delta}\left(\frac{|(\partial/\partial\bar{\zeta})\phi(\xi+i\eta)|^r\,d\eta\,d\xi}{|\eta|^{r/s}|\xi+i\eta-x|^{(p+1-(2/s))r}}\right)\left(\int_{-\infty}^{\infty}\int_{-\delta}^{\delta}\frac{|\eta|\,d\eta\,d\xi}{|\xi+i\eta-x|^2}\right)^{r/s}.\end{aligned}$$

We will use the fact that

$$\int_{-\delta}^{\delta}\int_{-\infty}^{\infty}\frac{|\eta|\,d\eta\,d\xi}{(\xi-x)^2+\eta^2} = 2\int_0^{\delta}\left(\int_{-\infty}^{\infty}\frac{d\xi/\eta}{((\xi-x)/\eta)^2+1}\right)d\eta = 2\pi\delta.$$

This implies

$$p!\left(\int_{-\infty}^{\infty}\left|\iint_{\psi}\frac{(\partial/\partial\bar{\zeta})\phi(\zeta)}{(\zeta-x)^{p+1}}\,d\zeta\wedge d\bar{\zeta}\right|^{r}dx\right)^{1/r}$$

$$\le (2\pi\delta)^{1/s}p!\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\int_{-\delta}^{\delta}\frac{\left|(\partial/\partial\bar{\zeta})\phi(\xi+i\eta)\right|^{r}}{|\eta|^{r/s}|\xi+i\eta-x|^{(p+1-2/s)r}}\,d\xi\,d\eta\right)dx\right)^{1/r}$$

$$= (2\pi\delta)^{1/s}p!\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\int_{-\delta}^{\delta}\left(\frac{\left|(\partial/\partial\bar{\zeta})\phi(\xi+i\eta)\right|^{r}}{|\eta|^{r/s}|\xi+i\eta-x|^{(p+1-2/s)r-2}}\right.\right.\right.$$

$$\left.\left.\left.\times\frac{d\xi\,d\eta}{|\xi+i\eta-x|^{2}}\right)dx\right)\right)^{1/r}$$

$$\le (2\pi\delta)^{1/s}p!\left(\int_{-\infty}^{\infty}\int_{-\delta}^{\delta}\left(\int_{-\infty}^{\infty}\left(\frac{\left|(\partial/\partial\bar{\zeta})\phi(\xi+x+i\eta)\right|^{r}dx}{|\eta|^{(p+1-1/s-2/r)r}}\right)\right.\right.$$

$$\left.\left.\times\frac{d\xi\,d\eta}{\xi^{2}+\eta^{2}}\right)\right)^{1/r}$$

$$\le (2\pi\delta)^{1/s}p!D_h\left(\int_{-\delta}^{\delta}\int_{-\infty}^{\infty}\left(\frac{|\eta|^{1/r}}{|\eta|^{p}e^{M^*(h/|\eta|)}}\right)^{r}\frac{d\xi\,d\eta}{\xi^{2}+\eta^{2}}\right)^{1/r}$$

$$\le D_h(2\pi\delta)^{1/s}h^{-p}M_p\left(\int_{-\infty}^{\infty}\int_{-\delta}^{\delta}\frac{|\eta|\,d\xi\,d\eta}{\xi^{2}+\eta^{2}}\right)^{1/r}\le AD_h\delta h^{-p}M_p.$$

Minkovski's inequality implies that for every $h>0$, resp. some $h>0$, there is a constant $C>0$ such that

$$\|\varphi^{(p)}\|_{L^r}\le CD_h h^{-p}M_p,\qquad p\in\mathbf{N}_0.$$

This implies the assertion.

Let f be a holomorphic function in $\psi_0\setminus\mathbf{R}$, where $\psi_0=\{z|(\operatorname{Im} z)<\delta_0\}$, $\delta_0=\delta_0(f)$. If for every $\varphi\in\mathscr{D}^*_{L^r}$ there exists the limit

$$\langle Tf,\varphi\rangle := \lim_{\varepsilon\to 0}\int_{\mathbf{R}}\varphi(x)(f(x+i\varepsilon)-f(x-i\varepsilon))\,dx,$$

then we call Tf the boundary value of f in $\mathscr{D}'^*_{L^r}$.

THEOREM 1. *Let f belong to $H^*_{L^s}$. Then for every $\varphi \in \mathscr{D}^*_{L^r}$,*

$$\langle Tf, \varphi\rangle = \int_{\psi} f(z)\frac{\partial}{\partial z}\phi(z)\,dz \wedge d\bar{z} - \int_{\Gamma_{\delta^-}} f(z)\phi(z)\,dz$$
$$+\int_{\Gamma_{\delta^-}} f(z)\phi(z)\,dz,$$

where φ is defined in Lemma 1. *Moreover, Tf belongs to $\mathscr{D}'^*_{L^r}$.*

Proof. Let

$$\psi_{a^+} = \{z; \operatorname{Im} z \in (0,\delta), |\operatorname{Re} z| < a\},$$
$$\psi_{a^-} = \{z; \operatorname{Im} z \in (-\delta, 0), |\operatorname{Re} z| < a\}, \qquad \delta \in (0, \delta_0),$$

and let $\varepsilon < (\delta_0 - \delta)/2$. Lemmas 2 and 3 enable us to apply Stokes' theorem which implies

$$\int\int_{\psi_{a^+}} f(x + i(y+\varepsilon))\frac{\partial}{\partial \bar{z}}\phi(z)\,d\bar{z}\wedge dz = \int_{\partial\psi_{a^+}} f(x+i(y+\varepsilon))\phi(z)\,dz.$$

Since $y + \varepsilon \in (\varepsilon, \varepsilon + \delta) \in (0, \delta_0)$ for $y \in (0, \delta)$, we obtain

$$\|f(\cdot + i(y+\varepsilon))\|_{L^s} \le Ce^{M^*(k/\varepsilon)}.$$

This fact and [13, p. 125, Lemma] imply $f(x + i(y+\varepsilon)) \to 0$ as $|x| \to \infty$, uniformly for $y \in (0, \delta)$. Thus, by Lemma 3 and by letting $a \to \infty$ we obtain

$$\int\int_{\psi_+} f(x+i(y+\varepsilon))\frac{\partial}{\partial\bar{z}}\phi(z)\,d\bar{z}\wedge dz$$
$$= \int_{-\infty}^{\infty} f(x+i\varepsilon)\phi(x)\,dx - \int_{-\infty}^{\infty} f(x+i(\varepsilon+\delta))\phi(x+i\delta)\,dx. \tag{8}$$

Similarly,

$$\int\int_{\psi_-} f(x+i(y-\varepsilon))\frac{\partial}{\partial z}\phi(z)\,d\bar{z}\wedge dz$$
$$= \int_{-\infty}^{\infty} f(x-i(\varepsilon+\delta))f(x-i\delta)\,dx - \int_{-\infty}^{\infty} f(x-i\varepsilon)\phi(x)\,dx. \tag{9}$$

We have (with suitable $C > 0$)

$$\begin{aligned}
&\left|\int\int_{\psi_+} f(x + i(y+\varepsilon))\frac{\partial}{\partial \bar z}\phi(z)\,d\bar z \wedge dz\right| \\
&\quad = 2\left|\int_0^\delta dy\left(\int_{-\infty}^{\infty} f(x + i(y+\varepsilon))\frac{\partial}{\partial \bar z}\phi(x+iy)\,dx\right)\right| \\
&\quad \le 2\int_0^\delta dy\left(\int_{-\infty}^{\infty} |f(x + i(y+\varepsilon))|^s\,dx\right)^{1/s} \\
&\qquad\quad \times\left(\int_{-\infty}^{\infty}\left(\left|\frac{\partial}{\partial \bar z}\phi(x+iy)\right|^r dx\right)\right)^{1/r} \\
&\quad \le C\int_0^\delta e^{M^*(k/(y+\varepsilon)) - M^*(k/y)}\,dy < \infty.
\end{aligned}$$

The same holds for the integral over ψ_-. Since the integrands in (8) and (9) pointwise converge to the corresponding integrable functions, as $\varepsilon \to 0$, we obtain

$$\langle Tf, \varphi\rangle = \int\int_{\psi} f(z)\frac{\partial}{\partial \bar z}\phi(z)\,d\bar z \wedge dz - \int_{\partial\psi} f(x)\phi(z)\,dz.$$

By using Hölder's inequality, the estimate for f, and Lemma 1 we obtain that for some $h > 0$ and some $C > 0$, resp. for every $h > 0$, there is $C > 0$, such that

$$|\langle Tf, \varphi\rangle| \le C\|\varphi\|_{h, L^r}.$$

For the proof of the next theorem we need the following estimate: There is $B > 0$ such that for every $y > 0$ and every $g \in L^s$ $(s > 1)$

$$\left(\int_{-\infty}^{\infty}\left|\int_{-\infty}^{\infty}\frac{g(t)\,dt}{t - x - iy}\right|^s dx\right)^{1/s} \le B\|g\|_s. \tag{10}$$

This estimate is obtained by combining Theorem 1.4, Lemma 1.5 (Ch. IV), Theorems 3.10 and 3.7 (Ch. II) in [11].

THEOREM 2. *The mapping* $T\colon H^*_{L^s} \to \mathscr{D}'^*_{L^s}$ *is surjective. Its kernel is* H_{L^s}.

Proof. We shall prove the assertion only for the (M_p)-case because the $\{M_p\}$-case can be proved similarly.

Let $f \in \mathscr{D}'^{(M_p)}_{L^s}$ be of the form (6). One can easily prove that the function $t \mapsto 1/(t - z)$, $t \in \mathbf{R}$, $z = x + iy$, $x \in \mathbf{R}$, $y \ne 0$, belongs to $\mathscr{D}^{(M_p)}_{L^r}$. We shall prove that

$$z \mapsto g(z) = -\left\langle f(t), \frac{1}{t - x - iy}\right\rangle, \qquad x \in \mathbf{R},\ y \ne 0,$$

belongs to $H_{L^s}^{(M_p)}$. By Minkovski's inequality and (10) we have

$$\begin{aligned}
&\left\| \left\langle f(t), \frac{1}{(t-\cdot)-iy} \right\rangle \right\|_{L^s} \\
&\qquad \le \left\| \sum_{p=0}^{\infty} \left\langle f_p(t), \frac{p!}{(t-z)^{p+1}} \right\rangle \right\|_{L^s} \\
&\qquad \le \left\| \left\langle f_0(t), \frac{1}{(t-\cdot)-iy} \right\rangle \right\|_{L^s} + \sum_{p=1}^{\infty} \left\| \left\langle f_p(t), \frac{p!}{(t-z)^{p+1}} \right\rangle \right\|_{L^s} \\
&\qquad \le B\|f_0\|_{L^s} + \sum_{p=1}^{\infty} \frac{p!}{y^{p-1}} \left\| \int_{\mathbf{R}} \frac{|f_p(t)|}{|t-z|^2}\, dt \right\|_{L^s}.
\end{aligned}$$

Since

$$\left(\int_{\mathbf{R}} \frac{dt}{|t-x-iy|^{1+r/2}} \right)^{s/r} = \frac{1}{|y|^{s/2}} \left(\int_{\mathbf{R}} \frac{du}{|u-i|^{1+r/2}} \right)^{s/r} = \frac{1}{y^{s/2}} A$$

$(r = s/(s-1))$, and for $p \ge 1$,

$$\left| \int_{\mathbf{R}} \frac{|f_p(t)|\, dt}{|t-z|^{3/2-1/r} |t-z|^{1/2+1/r}} \right|^s \le \int_{\mathbf{R}} \frac{|f_p(t)|^s}{|t-z|^{1+s/2}} \left(\int_{\mathbf{R}} \frac{dt}{|t-z|^{1+r/2}} \right)^{s/r},$$

we obtain

$$\begin{aligned}
&\left\| \left\langle f(t), \frac{1}{t-z} \right\rangle \right\|_{L^s} \\
&\qquad \le B\|f_0\|_{L^s} + A^{s/r} \sum_{p=1}^{\infty} \frac{p!}{|y|^{p-1+s/2}} \left(\int_{\mathbf{R}} \left(\int_{\mathbf{R}} \frac{|f_p(t)|^s}{|t-z|^{1+s/2}}\, dt \right) dx \right)^{1/s} \\
&\qquad \le B\|f_0\|_{L^s} + A^{s/r} \sum_{p=0}^{\infty} \frac{p!}{|y|^{p-1+s/2}} \left(\int_{\mathbf{R}} |f_p(t)|^s \right. \\
&\qquad\qquad\qquad \left. \times \left(\int_{\mathbf{R}} \frac{dx}{|t-x-iy|^{1+s/2}} \right) dt \right)^{1/s} \\
&\qquad \le B\|f_0\|_{L^s} + A^{s/r+1/s} \sum_{p=0}^{\infty} \frac{p!}{|y|^{p-1+s/2+1/2}} \|f_p\|_{L^s} \\
&\qquad \le A_1 \sum_{p=0}^{\infty} \frac{p!}{|y|^p} \|f_p\|_{L^s},
\end{aligned}$$

where $A_1 = B + A^{s/r+1/s}|y|^{(1-s)/2}$. This implies that for $y \neq 0$

$$\left\| \left\langle f(t), \frac{1}{t-z} \right\rangle \right\|_{L^s} \le A_1 \sup_p \left\{ \frac{k^p p!}{M_p |y|^p} \right\} \sum_{p=0}^{\infty} \frac{M_p}{k^p} \|f_p\|_{L^s} \le \tilde{A}_1 e^{M^*(k/|y|)},$$

and that $g \in H_{L^s}^{(M_p)}$. We shall show that $f = Tg$. Let $\varphi \in \mathscr{D}_{L^r}^{(M_p)}$ and let ϕ be its almost analytic extension. For $z \in \mathbf{C}$, put

$$\phi_1(z) = \frac{1}{2\pi i} \int_\psi \frac{(\partial/\partial\bar{\zeta})\phi(\zeta)}{\zeta - z} \, d\zeta \wedge d\bar{\zeta},$$

$$\phi_2(z) = \frac{1}{2\pi i} \int_{\Gamma_{\delta^-}} \frac{\phi(\zeta)}{\zeta - z} \, d\zeta, \qquad \phi_3(z) = -\frac{1}{2\pi i} \int_{\Gamma_{\delta^+}} \frac{\phi(\zeta)}{\zeta - z} \, d\zeta.$$

We have $\varphi(x) = \phi_1(x) + \phi_2(x) + \phi_3(x)$, $x \in \mathbf{R}$. By the same arguments as in Lemma 4 it follows that $x \mapsto \phi_2(x)$, $x \mapsto \phi_3(x)$, $x \in \mathbf{R}$, and in $\mathscr{D}_{L^r}^{(M_p)}$. Thus $x \mapsto \phi_1(x)$, $x \in \mathbf{R}$, is in $\mathscr{D}_{L^r}^{(M_p)}$. We have

$$\begin{aligned}
\langle f, \varphi \rangle = \frac{1}{2\pi i} \Bigg(& \left\langle f(x), \int_\psi \frac{(\partial/\partial\bar{\zeta})\phi(\zeta)}{\zeta - x} \, d\zeta \wedge d\bar{\zeta} \right\rangle \\
& + \left\langle f(x), \int_{\Gamma_{\delta^-}} \frac{\phi(\zeta)}{\zeta - x} \, d\zeta \right\rangle - \left\langle f(x), \int_{\Gamma_{\delta^+}} \frac{\phi(\zeta)}{\zeta - x} \, d\zeta \right\rangle \Bigg) \\
= \frac{1}{2\pi i} \Bigg(& \int_\psi \left\langle f(x), \frac{1}{\zeta - x} \right\rangle \frac{\partial}{\partial\bar{\zeta}} \phi(\zeta) \, d\zeta \wedge d\bar{\zeta} \\
& + \int_{\Gamma_{\delta^-}} \left\langle f(x), \frac{1}{\zeta - x} \right\rangle \phi(\zeta) \, d\zeta - \int_{\Gamma_{\delta^+}} \left\langle f(x), \frac{1}{\zeta - x} \right\rangle \phi(\zeta) \, d\zeta \Bigg) \\
= \int_\psi & g(\zeta) \frac{\partial}{\partial\bar{\zeta}} \phi(\zeta) \, d\zeta \wedge d\bar{\zeta} - \int_{\Gamma_{\delta^-}} g(\zeta)\phi(\zeta) \, d\zeta \\
& + \int_{\Gamma_{\delta^+}} g(\zeta)\phi(\zeta) \, d\zeta \\
= \langle & Tg, \varphi \rangle.
\end{aligned}$$

The interchange of f and integrals given above is allowed because one can prove that it is allowed if $\int_\psi$ and $\int_{\Gamma_{\delta\pm}}$ are replaced by $\int_{\psi_a}$ and $\int_{\Gamma_{a\delta\pm}}$, $a > 0$, and because

$$\int_{\psi_a} \frac{(\partial/\partial\bar\zeta)\phi(\zeta)}{\zeta-\cdot}\,d\zeta\wedge d\bar\zeta \to \int_{\psi} \frac{(\partial/\partial\bar\zeta)\phi(\zeta)}{\zeta-\cdot}\,d\zeta\wedge d\bar\zeta,$$

$$\int_{\Gamma_{a\delta\pm}} \frac{\phi(\zeta)}{\zeta-\cdot}\,d\zeta \to \int_{\Gamma_{\delta\pm}} \frac{\phi(\zeta)}{\zeta-\cdot}\,d\zeta, \qquad a\to\infty, \text{ in } \mathscr{D}_{L^r}^{(M_p)}.$$

By similar arguments as in the proof of Theorem 3.3 in [6] one can prove that Ker $T = H_{L^s}$.

3. BOUNDARY VALUES IN $\mathscr{D}'^*_{L^\infty}$ AND $\mathscr{D}'^*_{L^1}$

The method used in the previous section could not be applied for $s=\infty$ and $s=1$ because the function

$$\mathbf{R}\ni t\mapsto \frac{1}{t-x-iy}, \qquad x+iy\in\mathbf{C},\ y\neq 0,$$

is not in L^1. Note that this function belongs to $\dot{\mathscr{B}}^{(M_p)}$ but we did not succeed in proving that for an $f\in\mathscr{D}'^*_{L^\infty}$ or $f\in\mathscr{D}'^*_{L^1}$ there exists the corresponding $F(z)$ in $H^*_{L^\infty}$ or $H^*_{L^1}$ which converges to f in $\mathscr{D}'^*_{L^\infty}$ or $\mathscr{D}'^*_{L^1}$. We shall prove the converse assertion, i.e., that elements in $H^*_{L^\infty}$ and $H^*_{L^1}$ determine elements in $\mathscr{D}'^*_{L^\infty}$ and $\mathscr{D}'^*_{L^1}$ as boundary values but assuming the stronger condition (4) instead of (3). This condition enables us to follow the method of Komatsu [3, proof of Theorem 11.5]. The following lemma from [3] is needed.

LEMMA 5. *Let N_p satisfy* (1) *and* (3), $n_p = N_p/N_{p-1}$, *and let*

$$P(\zeta) = (1+\zeta)^2\prod_{p=1}^{\infty}\left(1+\frac{\zeta}{n_p}\right), \qquad \zeta\in\mathbf{C},$$

$$G(z) = \frac{1}{2\pi}\int_0^\infty P(\zeta)^{-1}e^{iz\zeta}\,d\zeta, \qquad z\in\mathbf{C}.$$

Then $G(z)$ is a holomorphic function which can be continued analytically to the Riemann domain $\{z;\ -\pi<\arg z<2\pi\}$ *on which we have* $P(D)G(z) = -(2\pi iz)^{-1}$. *$G(z)$ is bounded on the domain* $\{z; -(-\pi/2)\le \arg z\le 3\pi/2\}$. *Furthermore, set for $y>0$*

$$g(y) = G_+(-iy) - G_-(-iy),$$

where G_+ *is the branch of* G *on* $\{z;\ -\pi < \arg z \le 0\}$ *and* G_- *is that on* $\{z;\ \pi \le \arg z < 2\pi\}$. *Then for some* $A > 0$

$$|g(y)| \le A\sqrt{y}\, e^{M^*(L/y)}, \qquad y > 0.$$

THEOREM 3. *Assume* (1)–(4) *hold. Let* $F \in H^*_{L^\infty}$, *resp.* $F \in H^*_{L^1}$. *Then* (*in the sense of convergence in* $\mathscr{D}'^*_{L^\infty}$, *resp.* $\mathscr{D}'^*_{L^1}$,

$$F(x+iy) \to F(x+i0) \in \mathscr{D}'^*_{L^\infty}, \qquad y \to 0^+,$$
$$\textit{resp.}\ F(x+iy) \to F(x+i0) \in D'^*_{L^\infty}, \qquad y \to 0^+.$$

Proof. We shall prove the theorem only for the $\{M_p\}$-case since it is rather complicated. We shall use the construction from [3, Theorem 11.5] (see also [8]). Our aim is to prove that for any $\phi \in \mathscr{D}^{\{M_p\}}_{L^1}$, resp. $\phi \in \dot{\mathscr{B}}^{\{M_p\}}$, the set $\{\langle F(x+iy), \phi(x)\rangle;\ 0 < y < \delta_0\}$ is bounded and that for every $\phi \in \mathscr{D}^{\{M_p\}}$, $\langle F(x+iy), \phi(x)\rangle$ converges when $y \to 0$. Since $\mathscr{D}^{\{M_p\}}$ is dense in $\mathscr{D}^{\{M_p\}}_{L^1}$, resp. $\dot{\mathscr{B}}^{\{M_p\}}$, this will imply the assertion in Theorem 3.

Assume first that $F \in H^{\{M_p\}}_{L^\infty}$ and that $\phi \in \mathscr{D}^{\{M_p\}}_{L^1}$ such that for $h_0 > 0$, $\|\phi\|_{L^1, h_0} < \infty$.

Let $I_k = (k-2, k+2)$, $k \in \mathbf{Z}$, and ψ_k, $k \in \mathbf{Z}$, be a partition of unity in $\mathscr{D}^{\{M_p\}}$ such that for some $R > 0$, which does not depend on k,

$$\operatorname{supp} \psi_k \subset I_k, \qquad \|\psi_k\|_{L^1, h_0} \le R, \qquad k \in \mathbf{Z}.$$

There holds

$$\int_{-\infty}^{\infty} F(x+iy)\phi(x)\,dx = \sum_{k \in \mathbf{Z}} \int_{I_k} F(x+iy)\phi(x)\psi_k(x)\,dx, \qquad 0 < y < \delta_0.$$

We shall construct an ultradifferential operator of class $\{M_p\}$ of the form

$$P(D) = (1+D)^2 \prod_{p=1}^{\infty}\left(1 + \frac{D}{n_p}\right), \tag{11}$$

such that the equations

$$P(D)H_k(x+iy) = F(x+iy), \qquad k \in \mathbf{Z},$$

have the solutions $H_k(x+iy)$ which are holomorphic in

$$\Pi_k = \left\{x+iy;\ x \in I_k,\ 0 < y < \frac{\delta_0}{2}\right\}$$

and bounded in some neighborhood of I_k, $k \in \mathbf{Z}$.

As in [3, pp. 98–99], one can show that there is a sequence n_p such that the operator (11) is of class $\{M_p\}$, $M_p \prec N_p$, and

$$\| F(\cdot + iy) \|_{L^\infty} < Ce^{N^*(1/y)}, \qquad |y| < \delta_0.$$

Note that conditions (1), (2), and (4) imply that if $P(D)$ is of the form (11) then it is of the class $\{M_p\}$ and, for this, (4) could not be replaced by (3).

Fix k and denote by z_k^0 the point $k + i\delta$ $(\delta_0/2 < \delta < \delta_0)$. Let

$$H_k(z) = \int_\Gamma G(z - \omega) F(\omega)\, d\omega, \qquad z = x + iy,\, x \in I_k,\, 0 < y < \frac{\delta_0}{2},$$

where $G(z)$ is the Green kernel of $P(D)$ given in Lemma 5 and Γ_k is a simple closed curve laying in $\{x + iy;\, x \in I_k,\, y \in (0, \delta)\}$ starting at z_k^0 and encircling counterclockwise a slit connecting z_k^0 and z. We deform the path Γ_k to the union of segments joining z_k^0 and $z_k^1 = x + i(\delta_0/2)$, a segment joining z_k^1 and z, a segment joining z and z_k^1, and a segment joining z_k^1 and z_k^0. This is possible because $G(z)$ is bounded for $-\pi/2 \le \arg z \le 3\pi/2$. By the same arguments as in [3] we have $P(D)H_k(z) = F(z)$, $z \in \Pi_k$, and thus, we obtain

$$\begin{aligned} &\left| \int_{-\infty}^{\infty} F(x + iy) \phi(x)\, dx \right| \\ &\qquad \le \sum_{k \in \mathbf{Z}} \int_{I_k} |F(x + iy) \psi_k(x) \phi(x)|\, dx \\ &\qquad \le \sum_{k \in \mathbf{Z}} \int_{I_k} \left| \int_{\Gamma_k} G(z - \omega) F(\omega)\, d\omega P(D)(\psi_k(x)\phi(x)) \right| dx, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 < y < \frac{\delta_0}{2}. \end{aligned}$$

Denote the part of γ_k from z_k^1 to z and z to z_k^1 by Γ_k^1 and the rest by Γ_k^0, $k \in \mathbf{Z}$. We have

$$\begin{aligned} &\int_{I_k} \left| \left(\int_{\Gamma_k^1} G(z - \omega) F(\omega)\, d\omega \right) P(D)(\phi(x)\psi_k(x)) \right| dx \\ &\qquad \le \sup_{x \in I_k} \left\{ \left| \int_{\Gamma_k^1} G(z - \omega) F(\omega)\, d\omega \right| \right\} \int_{I_k} |P(D)(\phi(x)\psi_k(x))|\, dx. \quad (12) \end{aligned}$$

Denote by A_k the first and by B_k the second factor on the right side of (12). Since $P(D) = \sum_\alpha a_\alpha D^\alpha$ is of class $\{M_p\}$, from (5) with $2r < h_0^2$, and from $M_{\alpha-j}M_j \le M_\alpha$, $j \le \alpha$, j, $\alpha \in \mathbf{N}_0$, we have

$$\sum_{k\in\mathbf{Z}} B_k \le \sum_{k\in\mathbf{Z}} \sum_{\alpha=0}^{\infty} a_\alpha \sum_{j=0}^{\alpha} \binom{\alpha}{j} \int_{I_k} \left| \phi^{(\alpha-j)}(x)\psi_k^{(j)}(x) \right| dx$$

$$\le C \sum_{\alpha=0}^{\infty} \frac{r^\alpha}{(h_0^2)^\alpha} a_\alpha \sum_{j=0}^{\alpha} \binom{\alpha}{j} \|\phi\|_{L^1, h_0} \|\psi_k(x)\|_{L^\infty, h_0}$$

$$\le CR\|\phi\|_{L^1, h_0} \sum_{\alpha=0}^{\infty} \frac{r^\alpha}{(h_0^2)^\alpha} < \infty.$$

For A_k we have

$$A_k = \sup_{x\in I_k} \left\{ \left| \int_0^{\delta-y} g(t) F(x+iy+it)\, dt \right| \right\}$$

$$\le A\sqrt{y} \sup_{x\in I_k} \int_0^{\delta-y} e^{-N^*(1/(t+y))} e^{N^*(1/t)}\, dt < \infty.$$

This implies that $\sum_{k\in\mathbf{Z}} A_k B_k < \infty$. Consider the path Γ_k^0. We have

$$\left| \int_{I_k} \right| \left(\int_{\Gamma_k^0} G(z-\omega) F(\omega)\, d\omega \right) P(D)(\phi(x)\psi_k(x)) |dx|$$

$$\le \sup_{x\in I_k} \left\{ \left| \int_{\Gamma_k^0} G(z-\omega) F(\omega)\, d\omega \right| \right\} \int_{I_k} \left| P(D)(\phi(x)\psi_k(x)) \right| dx$$

$$= D_k B_k.$$

Since for $z \in \Pi_k$, $\omega \in \Gamma_k^0$, $G(z-\omega)$ is uniformly bounded by a constant which does not depend on k, we obtain $\sum_{k\in\mathbf{Z}} D_k B_k < \infty$. This implies

$$\left| \int F(x+iy)\phi(x)\, dx \right| \le \sum_{k\in\mathbf{Z}} (D_k + A_k) B_k < \infty.$$

The proof that there is $F(x+i0) \in \mathscr{D}'^{\{M_p\}}$ such that for every $\phi \in \mathscr{D}^{\{M_p\}}$

$$\langle F(x+iy), \phi(x) \rangle \to \langle F(x+i0), \phi(x) \rangle, \qquad y \to 0,$$

is given in [3, 6]. Thus, we conclude that

$$F(x+iy) \to F(x+i0) \in \mathscr{D}'^{\{M_p\}}_{L^\infty}, \qquad y \to 0.$$

The proof of Theorem 3 for $F \in H_{L^1}^{\{M_p\}}$ is analogous to the previous one. The partition of unity ψ_k and the constructed sequence $H_k(z)$, $z \in \Pi_k$, lead us to the proof that for every $\phi \in \mathscr{D}_{L^\infty}^{\{M_p\}}$

$$\left\{\langle F(x+iy), \phi(x)\rangle, 0 < y < \frac{\delta_0}{2}\right\}$$

is bounded. So we have to prove that for any $\phi \in \mathscr{D}^{\{M_p\}}$, $\langle F(x+iy), \phi(x)\rangle$ converges as $y \to 0^+$.

Let I be a bounded open interval and $\Pi_I = \{x + iy;\ x \in I,\ y \in (0, \delta_0/2)\}$. As in the first part of the proof we construct $P(D)$ of the form (12) and of $\{M_p\}$-class and H_1 such that $P(D)H_I(x+iy) = F(x+iy)$, $x + iy \in \Pi_I$. We put

$$H_I(x+iy) = \int_{\Gamma^1} G(z-\omega)F(\omega)\,d\omega + \int_{\Gamma^0} G(z-\omega)F(\omega)\,d\omega,$$

where $\Gamma = \Gamma^1 \cup \Gamma^2$ is a path constructed in the same way as Γ_k with I instead of I_k and $z^0 = x^0 + i\delta$ (x^0 is the middle point of I) instead of z_k^0. By using Hölder's inequality we obtain that $H(\cdot + iy) \in L^1(I)$ for every $0 < y < \delta_0/2$, and that

$$\| H(\cdot + iy)\|_{L^1} < C, \qquad 0 < y < \delta_0/2.$$

This implies that $H_I(x+iy) \to H_I(x+i0) \in L^1$, $y \to 0^+$, and thus, $H_I(x+iy) \to H_I(x+i0)$ in $\mathscr{D}'^{\{M_p\}}$.

It follows that for every $\phi \in \mathscr{D}^{\{M_p\}}$

$$\langle F(x+iy), \phi(x)\rangle \to \langle F(x+i0), \phi(x)\rangle, \qquad y \to 0,$$

and the proof is completed.

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